Debt policy in Euro-area countries: Evidence for Germany and Italy using penalized spline smoothing*

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Abstract

In this paper we test how the primary surplus in two countries of the Euro-area, Germany and Italy, reacts to changes of public debt. Our theoretical part demonstrates that a positive reaction on average gives strong evidence for a sustainable debt policy. In the empirical part, we perform semi-parametric estimations using penalized spline smoothing. The results suggest that in Germany debt policy has been sustainable, however, with a declining tendency. Italian public debt does not seem to be sustainable although consolidation efforts in the nineties have stabilized Italian debt.

JEL: H63, E62
Keywords: Government Debt, Intertemporal Budget Constraint, Penalized Splines, Semi-Parametric Estimation

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1 Introduction

In the last few years, Germany has been continuously violating the 3% deficit criterion of the Maastricht treaty, leading to a debt-GDP ratio of more than 65% in 2005. Italy could considerably reduce its deficits in the mid and late 1990s leading to a decline in its debt ratio. Nevertheless, the debt ratio still clearly exceeds 100% due to the excessive deficits in previous periods making it the largest debt ratio of the Euro-area countries. Further, most recently Italy again shows deficits exceeding 3% of GDP.

In the debate about deficit and debt ratios in countries of the Euro-area it has been argued that fiscal policy has been threatened to become unsustainable. Thus, a crucial issue about debt and deficits is the sustainability of public debt policy. Empirical studies which help to clarify whether governments pursue sustainable debt policies are indeed desirable and help to answer the question of whether policy makers adequately react to rising debt ratios.

For the United States a great many empirical studies exist beginning with the contribution by Hamilton and Flavin (1986). In their analysis they propose a framework for studying whether governments fulfill the intertemporal budget constraint and apply the tests to the United States for the time period from 1960-1984. Other papers followed which also investigated this issue for the United States, some of them confirming Hamilton and Flavin’s result while others reached different conclusions (see e.g. Kremers, 1988, Wilcox, 1989, or Trehan and Walsh, 1991). For European countries, Afonso (2005) gives a good survey of the studies about sustainability using time series methods.

However, these tests have been criticized by Bohn (1995, 1998) because they make assumptions about future states of nature that are difficult to estimate from a single set of observed time series data. In a recent paper Bohn (1998) proposes a new test where he suggests to empirically test whether the primary surplus to GDP ratio rises at least linearly with increases in the debt ratio. If the latter holds the intertemporal budget constraint of the government is fulfilled and public debt is sustainable.

In this paper we apply the test proposed by Bohn to Germany and Italy, the largest
and third largest country in the Euro-area, which have recently been characterized by large deficit and debt ratios as outlined above. Greiner et al. (2004) and Ballabriga and Marinez-Mongay (2005) have performed OLS estimations which study how the primary surplus to GDP ratio reacts to the debt ratio in some countries of the Euro-area. In Greiner et al. (2004a) evidence was found suggesting that the reaction of the primary surplus to variations in the debt ratio in Germany has not been constant over time. In this contribution, we extend these papers both from a theoretical as well as empirical point of view. So, we first derive necessary and sufficient conditions for a sustainable debt path assuming that the debt accumulation process is described by a stochastic differential equation. In a next step, we empirically test whether those conditions are satisfied for Germany and Italy by applying more sophisticated statistical methods.

In statistics a lot of work has been dedicated to the development of non-parametric estimation techniques during the last two decades with the book by Hastie and Tibshirani (1990) representing an initial milestone. Ruppert et al. (2003) give a good summary of the state of the art in this field and Kauermann (2005) presents a concise introduction to non- and semi-parametric estimation. With the software programme R and its estimation tools (see Wood, 2000, 2001) the technique can easily be applied to economics. Surprisingly, there exist only few economic applications of this estimation technique although it is clearly superior to OLS estimation. Contrary, in other sciences smoothing spline estimation has become somewhat a standard.

The rest of the paper is organized as follows. Section 2 gives some theoretical considerations concerning the primary surplus and the intertemporal budget constraint of the government. In particular, we consider a continuous time model and derive conditions for sustainability which are less restrictive than the ones in the literature. In Section 3 we present our estimation results for the two countries where we estimate time varying coefficients and where we apply a nonlinear estimation technique. Section 4, finally, concludes the paper.
2 The primary surplus and the intertemporal budget constraint

We consider a stochastic economy, where the accounting identity describing the accumulation of public debt in continuous time is given by the following stochastic differential equation:

\[ dB_t = (r(t, \omega)B_t - S(t, \omega))dt + \sigma dW_t, \]  

with \( B_t \) real public debt\(^1 \) at time \( t \), \( r(t, \omega) \) the random real interest rate and \( S(t, \omega) \) the real government surplus exclusive of interest payments on public debt. \( W_t \) is a Wiener process with constant volatility \( \sigma \) which is normalized to one, \( \sigma \equiv 1 \).

Next assume that the government in our economy chooses the primary surplus to GDP ratio, \( s(t, \omega) \), such that it is a positive linear function of the debt to GDP ratio, \( b(t, \omega) \), and of a time varying random term (cf. Bohn, 1995, 1998). The primary surplus ratio, then, can be written as

\[ s(t, \omega) = \alpha(t, \omega) + \beta(t)b(t, \omega), \]  

where \( \alpha(t, \omega) \) gives the random term and \( \beta(t) \) is the deterministic component determining how strong the primary surplus reacts to changes in the public debt ratio. \( \alpha \) can be interpreted as a systematic component showing how the level of the primary surplus reacts to a rise in GDP. \( \alpha \) may also contain other variables which affect the primary surplus ratio.

Multiplying both sides of (2) by the real GDP, denoted by \( Y(t, \omega) \), equation (2) can be rewritten as

\[ dB_t = (h(t, \omega)B_t - \alpha(t, \omega)Y(t, \omega))dt + dW_t, \]  

with \( h(t, \omega) \equiv r(t, \omega) - \beta(t) \). Solving equation (3) yields

\[ B_t = e^{\int_0^t h(\tau)d\tau} \left( B_0 - \int_0^t e^{-\int_0^\tau h(\mu)d\mu} \alpha(\tau)Y(\tau)d\tau + \int_0^t e^{-\int_0^\tau h(\mu)d\mu} dW_\tau \right), \]  

\(^1\)Strictly speaking, \( B_t \) should be real public net debt.
with $B_0$ public debt at time $t = 0$.

If the discounted government debt in equation (4) goes to zero in the limit, the government does not play a Ponzi game and we call a path of public debt which satisfies this constraint sustainable.

The next proposition gives necessary and sufficient conditions for the discounted value of (4) to converge to zero, that is for sustainability of public debt.

**Proposition**

Assume that the mean of the realized real interest rate is strictly positive for any interval $[t_1, t_2]$ sufficiently large. Then, we have the following result.

i) For $\alpha_t = 0$, \( \lim_{t \to \infty} \int_0^t \beta(\tau) d\tau = \infty \) is necessary and sufficient for \( e^{-\int_0^t r(\tau)d\tau} B_t \) to converge to zero.

(ii) For $\alpha_t \neq 0$, \( \lim_{t \to \infty} \int_0^t \beta(\tau) d\tau = \infty \) is necessary and

\[
\lim_{t \to \infty} \int_0^t \beta(\tau) d\tau = \infty \quad \text{and} \quad \lim_{t \to \infty} \frac{e^{-\int_0^t (r(\mu) - \gamma_y(\mu) - \gamma_\alpha(\mu)) d\mu}}{\beta(t)} = 0 \quad \text{with probability 1}
\]

are sufficient for \( e^{-\int_0^t r(\tau)d\tau} B_t \) to converge to zero, with $\gamma_y$ and $\gamma_\alpha$ the growth rate of $Y$ and $\alpha$, respectively.

**Proof:** See appendix.

The case (i) in the proposition, i.e. $\alpha_t = 0$, shows that $\beta$ may be negative for some time periods. Nevertheless, sustainability is given if it is positive on average as $t$ converges to plus infinity. For example, if $\beta$ is continuous and finite for $t \in (0, T]$, with $T > 0$, and constant and strictly positive for $t \in (T, \infty)$ the condition is trivially satisfied. It should be noted that $\alpha_t = 0$ implies that $\alpha$ is constant and equal to zero at any point in time.

Next, assume that $\alpha_t$ is either positive or negative and time varying, case (ii) in the proposition. Then, the proposition shows that $\lim_{t \to \infty} \int_0^t \beta(\tau) d\tau = \infty$ is necessary but may not be sufficient for the present value of public debt to converge to zero for $t \to \infty$.

But, if the interest rate exceeds the GDP growth rate plus the growth rate of the other factors affecting the primary surplus ratio on average, i.e. if $\int (r - \gamma_y - \gamma_\alpha) d\mu > 0$ holds,
the second condition for sufficiency is automatically fulfilled for $\lim_{t \to \infty} \int_0^t \beta(\tau) d\tau = \infty$. However, if the interest rate is below the GDP growth rate plus the growth rate of the other factors affecting the primary surplus ratio on average, sufficiency is only given if, in addition, $\beta(t)$ converges faster to plus infinity than $e^{-\int_0^t (r(\mu) - \gamma_y(\mu) - \gamma_\alpha(\mu)) d\mu}$. In economic terms the latter condition states that the reaction of the government to higher debt ratios must increase faster than the difference between the discount rate, the GDP growth rate and the growth rate of the other factors affecting the primary surplus ratio.

An important special case of (ii) is given when $\alpha_t$ is constant over time but not equal to zero. In this case, $\gamma_\alpha = 0$ holds and the difference between the real interest rate and the growth rate of GDP must be positive on average. This case is of interest because the difference between the real interest rate and the growth rate has been positive in most countries of the Euro area since the early 1980s. Then, a positive value for $\beta(t)$ on average is sufficient for a sustainable debt policy, provided that $\alpha(t)$ is constant.

It should also be noted that our proposition is the continuous time analogue of the discrete time model considered by Bohn (1998) and by Canzonerie et al. (2001), who took up the proof by Bohn (1998) and extended it to the case of a time varying $\beta$. The condition in our proposition differs from the one derived by Canzonerie et al. (2001) and is less restrictive. In Canzonerie et al. (2001) $\beta$ must be positive at each point in time among other things. In our approach $\beta$ may be negative for some periods, however, in the limit the integral must converge to plus infinity.

In the next section, we perform some empirical tests based on our theoretical considerations of this section in order to get evidence about sustainability of German and Italian debt policies. More precisely, we analyze whether estimating equation (2) yields positive values for $\beta(t)$ and we also estimate the time path of $\alpha(t)$.

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2Details can be found in the appendix.

3With the exception that both Bohn (1998) and Canzonerie et al. (2001) write the government budget constraint in a slightly different manner, namely as $B_{t+1} - B_t = r_{t+1} B_t - S_t (1 + r_{t+1})$. 

5
3 Empirical evidence for Germany and Italy

The previous section has presented an estimation strategy to test for sustainability of public debt. We here pursue this test and empirically analyze how the primary surplus to GDP ratio reacts to the debt ratio in order to see whether fiscal policies in Germany and Italy have been sustainable.

As outlined above the main idea in testing for sustainability is to estimate an equation of the following form,

\[ s_t = \beta_t b_t + \alpha^T Z_t + \epsilon_t, \]

where \( s_t \) and \( b_t \) are the primary surplus to GDP and the debt to GDP ratio, respectively. \( Z_t \) is a vector which consists of the number 1 and of other factors related to the primary surplus and \( \epsilon_t \) is an error term which is i.i.d. \( N(0, \sigma^2) \).

As concerns the primary surplus we use the surplus exclusive of the social surplus. We do this because social insurances do not borrow at the capital market. If the social insurances have a deficit and the government subsidizes social insurances which reduces the primary surplus, this amount appears as transfers and is included on the right hand side of equation (5). Further, if social insurances run surpluses these do not reduce public deficits since the government cannot use these surpluses which serve as reserves for social insurances.

As to the independent variables, or covariates, contained in \( Z_t \), which are assumed to affect the primary surplus, we include the social surplus to GDP ratio (\( Soc \)) and the real long-term interest rate (\( int \)). The social surplus ratio \( Soc \) affects the primary surplus when the government subsidizes social insurances, thus, reducing the primary surplus as just mentioned. Further, the social surplus ratio also serves as a business cycle indicator since it declines in recessions and rises in booms. The real interest rate is an important factor which determines the debt service and may have a positive or negative sign.

In addition, we posit that the debt ratio of the previous period \((t - 1)\) affects the primary surplus in \( t \) since we think that it is more appropriate to include lagged debt because government budget plans are usually made for the next year. Summarizing our
discussion, the equation to be estimated is as follows:

\[ s_t = \beta_t b_{t-1} + \alpha_0 + \alpha_1 Soc_t + \alpha_2 int_t + \epsilon_t. \]  

(6)

### 3.1 Germany

The data for Germany are annual data and cover the period from 1960 to 2003.\(^4\)

Fitting of model (6) is described in the appendix. In practice this can be carried out with the publicly available software R (http://www.r-project.org/) and the routine library mgcv (cf. Wood, 2000). The resulting estimate \( \beta_t \) is shown in figure 1. The dashed lines represent quantiles of bootstrap replicates, discussed in the appendix as well. We provide the 2.5%, 50% (median) and 97.5% quantiles, with the median given by the dashed line next to the point estimate which is given by the solid line. The remaining estimates are listed in table 1. The fitted residuals show no particular structure with a Durbin Watson statistic of 2.14.

Figure 1 shows that the reaction coefficient \( \beta_t \) has declined over time and the decline seems to have been stopped only in the mid 1990s. This means that over time the government has paid less attention to the question of sustainability of public debt. The reason for this may be that the scope of the government has continually decreased over time because the government had to finance other types of expenditures or simply that it was less willing to stabilize public debt. However, it can also be clearly realized that the coefficient has been positive over the whole period.

\(^4\)As to the data source, see Appendix B.
Figure 1: Time varying coefficient $\beta_t$ for Germany obtained from estimating (6).

<table>
<thead>
<tr>
<th>variable</th>
<th>estimate</th>
<th>std.err</th>
</tr>
</thead>
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<tr>
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<td>$0.015$</td>
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<tr>
<td>Soc</td>
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<td>$0.237$</td>
</tr>
<tr>
<td>int</td>
<td>$-0.060$</td>
<td>$0.167$</td>
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Table 1: Parameter estimates for Germany

The parameter for the social surplus is positive and statistically significant, while that for the interest rate is not significant. A positive sign of $\alpha_1$, the parameter giving the effect of the social surplus on the primary surplus, was to be expected, since the primary surplus is higher in good times when the government does not have to subsidize the social insurances. Further, as mentioned above the social surplus also serves as a business cycles indicator.
In order to relate our empirical analysis to the theoretical results derived in the above proposition, we consider $\alpha_t$ and the integral $\int_0^t \beta(\tau)d\tau$. We first consider the integral by replacing integration by summation $\sum \hat{\beta}(t)$ keeping in mind that we have equidistant observed points in time. The upper part of figure 2 shows the resulting fitted integral $\int \beta(\tau)d\tau$ with confidence band obtained by bootstrapping. Again the three lines correspond to the 2.5%, 50% and 97.5% quantiles of the bootstrapped estimate of the integral. There is clear evidence that the integral is increasing.

The lower part in figure 2 shows the estimated $\alpha_t$ which is given by $\alpha_t = \alpha_0 + \alpha_1 Soc_t + \alpha_2 int_t$. It should be noted that $\alpha_t$ is negative but time variation is ignorable, that is $\alpha_t$ does not vary with time other than with covariates. In addition, the difference between the long-term interest rate and the growth rate of GDP was on average 0.8 percent in Germany from 1960-2003. In particular, it has been strictly positive since 1980, with the exception of 1990 and 1991 when Germany was unified. Thus, for Germany case (ii) of our proposition applies. Since the growth rate of $\alpha_t$ is equal to zero and since $\beta_t$ is positive on average with a rising integral over time, German debt policy seems to be sustainable, provided this policy goes on infinitely one should add. But it must also be repeated that there is a clear negative trend in the coefficient $\beta_t$ which gives the response of the governments to variations in the debt ratio.
In the next subsection we perform these tests for Italy.

3.2 Italy

The data for Italy are annual records and we consider the period from 1975 to 2003. Data before 1975 were excluded because the data show extreme values which would act as leverage points in the estimation.

We therefore decided to concentrate on the reduced time window only. Fitting model (6) to Italian data gives the coefficient $\beta_t$ as shown in figure 3. As above the solid line is the point estimate and the dashed line next to it is the median. The upper and lower dashed lines are again the 2.5 and 97.5% quantiles which were obtained by bootstrapping. The figure shows that the point estimate for the coefficient has been negative up to the...
early 1990s when it became positive. Most recently, however, the coefficient seems to decline again. The positive trend in the coefficient was probably generated by the need to fulfill the Maastricht criteria in order to join the Euro area. But at no point in time the estimate $\hat{\beta}_t$ is significantly different from zero. The parametric estimates are listed in table 2.

![Figure 3: Time varying coefficient $\beta_t$ for Italy obtained from estimating (6).](image)

<table>
<thead>
<tr>
<th>variable</th>
<th>estimate</th>
<th>std.err</th>
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<tbody>
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<td>0.063</td>
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<tr>
<td>Soc</td>
<td>0.856</td>
<td>0.360</td>
</tr>
<tr>
<td>int</td>
<td>$-0.156$</td>
<td>0.126</td>
</tr>
</tbody>
</table>

Table 2: Parameter estimates for Italy

As for Germany the coefficient $\alpha_1$, giving the effect of the social surplus on the primary
surplus, is positive and statistically significant, which makes sense from an economic point view as already set forth above. The estimate for the interest rate is insignificant as for Germany, too.

In the upper part of figure 4 we show the integral $\int \beta(\tau) d\tau$ which has been negative throughout the sample period.

![Integral Plot](image)

![Intercept Plot](image)

Figure 4: Time varying coefficient $\beta_t$ for Italy obtained from estimating (6).

The lower part of figure 4 gives the estimated values for $\alpha_t$ which is given by $\alpha_t = \alpha_0 + \alpha_1 Soc_t + \alpha_2 int_t$. The hypotheses that $\alpha_t$ is equal to zero and constant over time clearly cannot be rejected. Further, the difference between the long-term interest rate and the GDP growth rate was positive on average in Italy over the time period we consider, so that condition (i) in the above proposition applies.

Thus, a positive value of $\beta_t$ on average would be necessary and sufficient for sustainability of fiscal policy provided that the past gives a representative picture. However, as figure 3 and the upper part of figure 4 show, Italian public debt policy clearly does not appear to be sustainable although there has been a positive trend in the response of the primary surplus to the debt ratio. The integral is not significantly different from zero
throughout the whole period with a negative point estimate. This implies that there is no significant evidence that Italian public debt policy is sustainable.

4 Conclusion

It has been argued that sustainability of fiscal policy cannot be tested unless strong assumptions about the distribution of government debt in the future are made (Bohn, 1995, p. 269). This seems to be correct since the intertemporal budget constraints states that discounted public debt must converge to zero only asymptotically. Nevertheless, studying the behaviour of debt policy of governments in the past gives valuable information as to how governments cope with rising government debt and allows to answer the question of whether debt policy has to change in order to assure sustainability of public debt.

From a theoretical point of view we have demonstrated that a necessary condition for sustainability of public debt in a stochastic environment, is a reaction coefficient of the primary surplus to higher debt ratios which is positive on average. Thus, a positive reaction coefficient on average provides strong evidence for a sustainable debt policy.

This paper has then analyzed public debt policy in Germany and Italy over the last 30 to 40 years. We took these two countries because they belong to the largest countries in the Euro area and because they have been characterized by large public deficits recently and by high debt ratios, respectively. For Germany, our estimations suggest that the primary surplus to GDP ratio is a positive function of the debt-GDP ratio on average. However, the response has steadily declined over the time. For Italy, the response has been negative up to the early 1990s when the point estimate of the coefficient become positive but declined again most recently.

Making strong assumptions about the future and assuming that our observations give a representative picture of the two countries considered, our analysis shows that public debt seems to be sustainable in Germany but not in Italy. The reason is that the coefficient determining about sustainability of public debt has been positive, on average for Germany while it turned out to be negative for Italy.
A Proof of the proposition in the text

Multiplying both sides of (4) by the discount factor $e^{-\int_0^t r(\tau) d\tau}$ and rewriting gives

$$
e^{-C_3(t)}B_t = e^{-C_1(t)}B_0 - \alpha_0 Y_0 e^{-C_1(t)} \int_0^t e^{C_1(\tau)} e^{C_2(\tau)} e^{-C_3(\tau)} d\tau + e^{-C_1(t)} \int_0^t e^{C_1(\tau)} e^{-C_3(\tau)} dW_\tau, \tag{7}$$

with

$$\int_0^t \beta(\tau) d\tau \equiv C_1(t), \int_0^t \beta(\mu) d\mu \equiv C_1(\tau), \int_0^T (\gamma_y(\mu) + \gamma_0(\mu)) d\mu \equiv C_2(\tau), \int_0^t r(\mu) d\mu \equiv C_3(\tau),$$

where $\gamma_y$ and $\gamma_0$ give the growth rate of $Y$ and $\alpha$, respectively.

First, we show case (ii). Necessity is immediately seen. In order to prove sufficiency, $\lim_{t \to \infty} C_1(t) = \lim_{t \to \infty} \int_0^t \beta(\tau) d\tau = \infty$ must hold so that the first term on the right hand side in (7), i.e. $e^{-C_1(t)} B_0$, converges to zero.

The second term on the right hand side in (7) can be written as

$$\int_0^t \frac{e^{C_1(\tau)} e^{C_2(\tau)} e^{-C_3(\tau)} d\tau}{e^{C_1(t)}} \equiv D(t), \tag{8}$$

where we have normalized $\alpha_0 Y_0 \equiv 1$.

If $\int_0^\infty e^{C_1(\tau)} e^{C_2(\tau)} e^{-C_3(\tau)} d\tau$ remains bounded $\lim_{t \to \infty} C_1(t) = \infty$ guarantees that $D$ converges to zero. Boundedness of $\int_0^\infty e^{C_1(\tau)} e^{C_2(\tau)} e^{-C_3(\tau)} d\tau$ is given for $\lim_{t \to \infty} (C_1(t) + C_2(t) - C_3(t)) = -\infty$ with probability 1. If $\lim_{t \to \infty} \int_0^t e^{C_1(\tau)} e^{C_2(\tau)} e^{-C_3(\tau)} d\tau = \infty$ with probability 1, applying l’Hôpital gives the limit of $D$ as

$$\lim_{t \to \infty} D(t) = \lim_{t \to \infty} \frac{e^{C_2(t)} e^{-C_3(t)}}{\beta(t)}. \tag{9}$$

This shows that $D$ converges to zero in the limit if the condition in the proposition is fulfilled.

The third term on the right hand side in (7) is stochastic with the expected value equal to zero. Defining the third term as $X_t(\omega) \equiv e^{-C_1(t)} \int_0^t e^{C_1(\tau)} e^{-C_3(\tau)} dW_\tau(\omega)$, the second moment can be written as

$$E[X_t^2(\omega)] = E \left[ \left( \int_0^t \frac{e^{C_1(\tau)} e^{-C_3(\tau)} dW_\tau(\omega)}{e^{C_1(t)}} \right)^2 \right] d\tau = \left( \int_0^t E \left[ \frac{e^{2C_1(\tau)} e^{-2C_3(\tau)}}{e^{2C_1(t)}} \right] d\tau \right), \tag{10}$$
because $E \left[ \left( \int_0^t e^{C_1(\tau)} e^{-C_3(\tau)} dW_\tau(\omega) \right)^2 \right] = \int_0^t E \left[ \left( e^{C_1(\tau)} e^{-C_3(\tau)} \right)^2 \right] d\tau$.

Since the mean of the realized real interest rate is strictly positive, we can find a constant $\bar{r} > 0$ so that $-C_3(\tau) = -\int_0^\tau r(\mu) d\mu \leq -\int_0^\tau \bar{r} d\mu$. Then, we can write

$$\frac{\int_0^t e^{2C_1(\tau)} E \left[ e^{-2C_3(\tau)} \right] d\tau}{e^{2C_1(t)}} \leq \frac{\int_0^t e^{2C_1(\tau)} e^{-2\bar{r} \tau} d\tau}{e^{2C_1(t)}}. \quad (11)$$

If $\int_0^t e^{2C_1(\tau)} e^{-2\bar{r} \tau} d\tau$ remains bounded and if $\lim_{t \to \infty} C_1(t) = \infty$ holds, the expression in (10) converges to zero. If $\int_0^t e^{2C_1(\tau)} e^{-2\bar{r} \tau} d\tau$ diverges, applying l'Hôpital gives the r.h.s. in (11) as $e^{-2\bar{r} t} / 2\beta(t)$ showing that $\beta(t)$ must not converge to zero faster than $e^{-2\bar{r} t}$ if (11) is to converge to zero asymptotically. Now, assume that $\beta(t)$ declines exponentially. This would imply that $\lim_{t \to \infty} C_1(t) = \lim_{t \to \infty} \int_0^t \beta(\tau) d\tau < \infty$ holds. Consequently, if $\lim_{t \to \infty} C_1(t) = \infty$ holds, $\beta(t)$ cannot decline exponentially, and (11) converges to zero, too.

Case (i) is immediately seen by setting $\alpha_0 = 0$ in equation (7). This proves the proposition. \qed

B Data

Source: OECD Economic Outlook Statistics and Projections

We use the Data Set corresponding to those published in the June 2003 issue of the OECD Economic Outlook. Especially, we take the entire Data set for the Government Account and the series for Gross Domestic Product at Market prices (GDP)

C Nonparametric estimation

The subsequent algorithm is based on Wood (2000) and implemented in the public domain software package R (see Ihaka & Gentleman, 1996). The program and more information about it can be downloaded from http://www.r-project.org/. We exemplify the fit with
the simplified model

\[ s_t = \alpha_0 + \beta_t b_{t-1} + \varepsilon_t \]

Let \( s_t \) and \( b_{t-1} \) be the observed values for \( t = 2, 3, \ldots \). For fitting we replace the functional shape \( \beta_t \) by the parametric form

\[ \beta_t = \beta_{00} + \beta_{01} t + Z(t)\gamma \]  

(12)

where \( Z(t) \) is a high dimensional basis in \( t \). A typical setting is to set \( Z(t) \) as cubic spline basis with knots chosen at the observed points in time. However, numerically more efficient is to work with a reduced basis as suggested in O’Sullivan (1986) or Wood (2000). The latter proposal is implemented in R. The idea is to construct only those basis functions corresponding to the largest eigenvalues of \( Z(t)Z(t)^T \) (see Wood (2000) for more details).

In principle, with replacement (12) one ends up with a parametric model. However, fitting the model in a standard OLS fashion is unsatisfactory due to the large dimensionality of \( Z(t) \) which will lead to highly variable estimates. This can be avoided by imposing an additional penalty term on \( \gamma \), shrinking its values to zero. To be more specific, we obtain an estimate by minimizing the penalized OLS criterion

\[ \sum_t \left\{ s_t - d_t \beta_d - Z(d_t)\gamma \right\}^2 + \lambda \gamma^T P \gamma \]

with \( \lambda \) called the smoothing or penalty parameter and \( \gamma^T P \gamma \) as penalty. Matrix \( P \) is thereby chosen in accordance to the basis and for cubic splines the penalty corresponds to the integrated square derivative of \( \beta_t \) (see also Ruppert, Wand & Carroll, 2003, for more details). It is easy to see that choosing \( \lambda = 0 \) yields an unpenalized OLS fit, while \( \lambda \to \infty \) typically implies \( \gamma = 0 \) depending on the choice of \( P \). Hence, \( \lambda \) steers the amount of smoothness of the function with a simple linear fit as one extreme and a high dimensional parametric fit as the other extreme.

Let \( \beta = (\beta_1, \beta_2, \ldots)^T \) be the time varying effect stacked up to a column vector and assume for simplicity of presentation that \( \alpha_0 \equiv 0 \). Let \( t \) be the vector of observed points in time and \( Z(t) \) the spline basis evaluated at these points. With the spline approximation we set \( \beta = B(t)\theta \) where \( B(t) = (1, t, Z(t)) \) and \( \theta = (\beta_{00}, \beta_{01}, \gamma)^T \). The estimate \( \hat{\beta} \), say,
is then available in analytic form via \( \hat{\beta} = H(\lambda)s \), with \( s = (s_1, s_2, \ldots)^T \) and \( H(\lambda) \) as hat or smoothing matrix, respectively, defined through

\[
H(\lambda) = B(t) \left( B^T(t)B(t) + \lambda \begin{pmatrix} 0 & 0 \\ 0 & P \end{pmatrix} \right)^{-1} B^T(t)
\]

Note that \( H(0) \) and \( H(\infty) \) are classical hat matrices while \( H(\lambda) \) for \( 0 < \lambda < \infty \) is a penalized version. The trace of \( H(\lambda) \) is usually understood as the degree of the fit where \( 2 = \text{tr}(H(\infty)) \leq \text{tr}(H(\lambda)) \leq \text{tr}(H(0)) = p + 2 \) with \( p \) as dimension of \( Z(t) \). The linear operator also allows to easily calculate variances of the estimate via

\[
\text{Var} \left( \hat{\beta} \right) = H(\lambda)\Sigma(s)H^T(\lambda)
\]

with \( \Sigma(s) \) as covariance matrix of \( s \). Assuming uncorrelated and homoscedastic residuals (in our data example autocorrelation was not found) we get \( \text{Var} \left( \hat{\beta} \right) = \hat{\sigma}^2 H(\lambda)H^T(\lambda) \) with \( \hat{\sigma}^2 \) as residual variance estimate. Finally, if additional covariates are in the model, like in (6), we pursue the same estimation like above but with hat matrix \( H(\lambda) \) being supplemented by the additional covariate vectors.

To obtain a reliable fit, \( \lambda \) should be chosen data driven. One possibility is to use a generalized cross validation criterion defined through

\[
GCV(\lambda) = \sum_t \left( \frac{s_t - \alpha_0 - \hat{\beta}_t b_{t-1}}{1 - \text{tr}(H)/n} \right)^2
\]

with \( n \) as overall sample size. A suitable choice for \( \lambda \) is achieved by minimizing \( GCV(\lambda) \). This can be done iteratively using a Newton-Raphson algorithm, as has been pointed out and implemented by Wood (2000, 2001).

\section*{D Bootstrapping}

Our intention is to assess estimation variability of a statistics \( T \), say, based on observations based on model (6). For instance statistics \( T \) could be the nonparametric estimate \( \hat{\beta}_t \) itself or its integrated version \( \int_t \hat{\beta}(\tau)d\tau \). Variability occurs due to residual \( \epsilon_t \) which is
bootstrapped from the fitted residuals $\hat{\varepsilon}_t = s_t - \hat{\mu}_t$ with $\hat{\mu}_t$ as fitted mean value obtained from model (6). Bootstrapping is thereby a convenient method to assess estimation variability. The basic ideas are discussed in Efron & Tibshirani (1993), Shao & Tu (1995) or Davison & Hinkley (1997). We here pursue a wild bootstrap (see e.g. Mammen, 1993) which captures heteroscedasticity as well. The idea is to resample $\hat{\varepsilon}^*_t$ from the two points $\hat{\varepsilon}_t(1/2 + \sqrt{5}/10, 1 + \sqrt{5}/10)$ with probabilities $\pi$ and $1 - \pi$ with $\pi = 1/2 + \sqrt{5}/10$. This approach reproduces $E^* (\hat{\varepsilon}^*_t) = 0$, $E^* (\hat{\varepsilon}^*_t^2) = \bar{\varepsilon}^2$ and $E^*(\hat{\varepsilon}^*_t^3) = \bar{\varepsilon}^3$ with $E^*(\cdot)$ indicating the mean over the bootstrap sampling scheme. The bootstrap replicate $\hat{\varepsilon}^*$ produces the bootstrap observations

$$s^*_t = \hat{\mu}_t + \hat{\varepsilon}^*_t$$

which are then used to refit model (6) to the bootstrapped data. This in turn leads to a bootstrap replicate $T^*$ of the focussed statistics. Repeating this step $B$ times provides a bootstrap approximation for the sample distribution of $T$. 
References


